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Towards a description of complexity of the simplest cosmological systems

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Abstract. We study the problem of complex dynamics and nonintegrability (integrability) of cosmological dynamical systems which are given in the Hamiltonian form with indefinite kinetic energy form $T = \frac{1}{2}g(v, v)$ where g is a two-dimensional pseudo-Riemannian metric with a Lorentzian signature $(+, -)$, and $v \in T_x\mathcal{M}$ is a tangent vector at a point $x \in \mathcal{M}$ of the configuration space $\mathcal{M}$. We present examples showing the effectiveness of using (a) the direct method of construction of linear and quadratic first integrals, (b) the Ziglin and Yoshida theorems concerning nonintegrability of Hamiltonian systems with homogeneous potential functions and (c) the Morales–Ramis theorem of nonintegrability of Hamiltonian systems with complex potential functions.

1. Introduction

Relativistic cosmological models are based on the Einstein theory of gravitation. In this theory gravitation is interpreted as a kind of deformation of spacetime due to the presence of matter and energy in the space. The Einstein field equations describe the dynamical evolution of spacetime as well as the motion of matter and physical fields. They constitute a system of nonlinear partial differential equations. Without some simplifying assumptions they are intractable by analytical methods. The physically motivated assumption is to postulate a certain symmetry of spacetime; for example, that it has the topology of $\mathbb{R}\times$ {space with the Friedman–Robertson–Walker metric}. Usually, such an idealization allows us to reduce the Einstein field equation to a system of ordinary differential equations. Therefore it seems to be natural to adopt the dynamical systems theory to analyse the evolution of spacetime. It frequently happens that the reduced system is a Hamiltonian one with $\mathbb{R}^{2n}$ (equipped with the standard symplectic structure) as the phase space. The Hamiltonian function of this system in many cases is given by the following formula:

$$H(q, p) := T(q, p) + V(q) \qquad (q, p) \in \mathbb{R}^{2n} \tag{1}$$

where the kinetic energy

$$T(q, p) = g^{\alpha\beta} p_\alpha p_\beta \qquad (\alpha, \beta = 1, \ldots, n)$$

is not positively defined. This form is close to the well known form of a natural mechanical system, and thus one can consider direct applications of methods developed for studying the dynamics of such systems. Several topics seem to be very attractive. For example, it is important to determine conditions for integrability and separability. Rabinowitz [23, 24] used

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the methods of modern theory of critical points for indefinite functionals in an analysis of periodic orbits for Hamiltonian systems. These methods can be adopted in our case. One can also look for effective tools for proving nonintegrability and the presence of chaos in such a system. However, the fact that system (1) looks similar to a natural mechanical system does not help much for several reasons of two origins. Of course, to study such a system we can always apply an arbitrary method for which the particular form of the Hamiltonian function is irrelevant. However, the cosmological origin of the system imposes certain constraints of physical origin and this needs some additional nontrivial investigation. For example, assume that we study the nonintegrability of system (1) and we are able to prove that it is not integrable (e.g. in the Liouville sense). Usually, for the vacuum cosmological models only level $\Omega = \{(q, p) \in \mathbb{R}^{2n} | H(q, p) = 0\}$ has a physical interpretation, and a system nonintegrable in the whole phase space can be integrable in Ω. Thus, our answer concerning nonintegrability (or even the nonexistence of one additional first integral) has no significant physical meaning.

On the other hand, the form of the system (1) suggests reducing its dynamics to the region of possible motion in the configuration space. For natural mechanical systems there exist effective methods for study of their dynamics by means of the variational approach. As far as we know, it is not known whether these methods can be extended in such a way that we can apply them to study systems with indefinite kinetic energy forms. There exist beautiful theorems which connect the nonintegrability of a natural mechanical system with the topology of its configuration space [21]. Thus we can formulate the following problem. Let $\mathcal{M}$ be a pseudo-Riemannian manifold with metric $g_{\alpha,\beta}$, and (q, p) be local coordinates on $T^*\mathcal{M}$. We consider system (1) on $T^*\mathcal{M}$. A natural question arises: do restrictions on the topology of $\mathcal{M}$ exist (e.g. on its Euler characteristic) preventing the integrability of system (1) [21]? It would also be interesting to consider the problems of separability in this context.

Indefinite Hamiltonians appear also in biological models arising in ecology. Rod and Sleeman [27] show that these Hamiltonian systems have chaos in the sense that there are nondegenerate homoclinic or heteroclinic solutions connecting hyperbolic periodic orbits. Then near the transversal intersection of the stable and unstable manifolds of these hyperbolic periodic orbits one can embed a Smale horseshoe map into the dynamics. This proves directly that the flow has no real analytic integral independent of the Hamiltonian. As was pointed out by Rod and Sleeman, the basic tool for this approach was developed by Ziglin in the context of complex analytic Hamiltonian systems [27].

It should be mention here that Hofer and Toland [17] proved some general theorems concerning the existence of periodic and homoclinic and heteroclinic orbits for a wide class of indefinite Hamiltonian systems provided that some assumptions about the behaviour of a potential function are satisfied. A kinetic energy form is indefinite but not degenerate. They proved theorems on a certain subset of zero energy ($\mathcal{H} = 0$) which are met in cosmological applications. The problems in this class arise in nonlinear mechanics and the underlying motivation arises from modelling nonlinear water waves [29].

The main aim of this paper is to demonstrate how standard methods of investigation of integrability (nonintegrability) work when applied to studying systems of cosmological origin. The example of our discussion is a class of two-dimensional Hamiltonian Friedmann–Robertson–Walker (FRW) models with scalar fields (see table 1). In section 2 the notation and background for further investigation is given. Section 3 contains the derivation of the FRW dynamical system. In section 4 we present results obtained by direct construction of integrable systems. Section 5 contains application of the Ziglin theory and its modification for proving the nonintegrability of the FRW system.

Table 1. Examples of simple relativistic dynamical systems of two-dimensional configuration space. In these simple, low-dimensional systems we find or suspect complex (chaotic) behaviour of trajectories in the phase space. Two identical trajectories of the system starting at slightly different positions (initial conditions) diverge in time. Such sensitive dependence on initial conditions is the main characteristic of chaotic systems and means that they are difficult to predict over long timescales, practically over the Lapunov characteristic time [30].

Hamilton function	Remarks
FRW cosmology coupled to real free massive scalar field $\mathcal{H} = \frac{1}{2}(-p_1^2 + p_2^2) + \frac{1}{2}(-q_1^2 + q_2^2 + m^2 q_1^2 q_2^2) = 0$	$m =$ constant [3,28]
Single scalar field evolving in the idealized de Sitter space $\mathcal{L} = \mathrm{e}^{3\nu t}\left[\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\mathrm{e}^{-2\nu t}(\nabla\Phi)^2 + \frac{1}{2}\mu^2\Phi^2 - \frac{1}{4}\lambda\Phi^4 - \frac{\mu^2}{4\lambda}\right]$	$\Phi(x, y, z, t)$, scalar field [16]
FRW model with conformally coupled massive, real, self-interacting scalar field $\mathcal{H} = \frac{1}{2}\left[-(p_1^2 + kq_1^2) + (p_2^2 + kq_2^2) + m^2 q_1^2 q_2^2 + \frac{\lambda}{2}q_2^4 + \frac{\Lambda}{2}q_1^4\right] \equiv 0$	$\Lambda, \lambda, m =$ constant, $k = 0, \pm 1$ [1]
Bianchi IX model with two scale function q_A, q_B, dust and cosmological constant $\mathcal{H} = \frac{p_A p_B}{4B} - \frac{q_A p_A^2}{8q_B^2} + 2q_A - \frac{q_A^3}{2B^2} - 2\Lambda q_A q_B^2 - E_0 = 0$	Λ, cosmological constant [9]
FRW model with cosmological constant and a conformally coupled scalar field with the potential $V(\Phi) = \Phi^2 m^2 + \frac{\lambda}{4}\Phi^4$ $\mathcal{L} = \frac{1}{2}\left\{-\dot{a}^2 + ka^2 + \dot{\psi}^2 - k\psi^2 - m^2\psi^2 a^2 - \frac{\bar{\lambda}}{2}\Phi^4 - \frac{\bar{\Lambda}}{2}a^4\right\}$	$\psi \propto a\Phi$, rescaled scalar field a, scale factor m, mass of field
FRW model with cosmological constant and a conformally coupled scalar field with the potential $V(\Phi) = V\left(\frac{\psi}{a\sqrt{v}}\right)$ (flat case with $k = 0$) $\mathcal{H} = -\frac{1}{2}p_a^2 + \frac{1}{2}p_\psi^2 + \frac{\Lambda}{4}a^4 + va^4 V\left(\frac{\psi}{a\sqrt{v}}\right)$	$v = \int_V \sqrt{g^3}\,\mathrm{d}\chi\,\mathrm{d}\theta\,\mathrm{d}\phi$ $= \int_V \sqrt{g^3}\,\mathrm{d}^3 x$ [5]

2. Simple indefinite Hamiltonian systems

Let us consider a simple mechanical system (SMS) described by a natural Lagrangian

$$\mathcal{L} = \tfrac{1}{2}g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - V(q) \qquad \alpha, \beta = 1, \ldots, n \tag{2}$$

where potential $V : \mathcal{M} \to \mathbb{R}$ is a smooth function (C^∞ class) defined on a configuration space $\mathcal{M}$ with q^α being local generalized coordinates and $\dot{q}^\alpha = \frac{\mathrm{d}}{\mathrm{d}t}q^\alpha$ their generalized velocities; $g_{\alpha\beta}$ are components of a symmetric metric tensor on $\mathcal{M}$.

The Hamiltonian of an SMS has the form

$$H = \tfrac{1}{2}g^{\alpha\beta}p_\alpha p_\beta + V(q) \tag{3}$$

where $p_\alpha = g_{\alpha\beta}\dot{q}^\beta$ are canonical momenta. This function is a first integral of motion. Trajectories of an SMS of prescribed energy E are located in a subset $\Omega \subset T\mathcal{M}$ of the phase space defined by the level of constant value of the Hamiltonian

$$\Omega = \{(q, \dot{q}) \in T\mathcal{M} : g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = 2(E - V)\}. \tag{4}$$

When the kinetic energy $T = \frac{1}{2}g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta$ is positive definite the motion of a system takes place in the domain

$$\mathcal{D} = \{q \in \mathcal{M} : E - V(q) \geqslant 0\}$$

of the configuration space. Generally, this subset has a nonempty boundary

$$\partial\mathcal{D} := \{q \in \mathcal{M} : E - V(q) = 0\}. \tag{5}$$

In the case of relativistic systems of general relativity and cosmology it is typical that the kinetic energy form is indefinite. Therefore such systems are called simple indefinite mechanical systems (SIMSs). Systems for which the kinetic energy is positive definite we call simple classical mechanical systems (SCMSs).

There is one main difference between SCMSs and SIMSs. In the last case the whole accessible configuration space can consist of two regions with positive and negative kinetic energy separated by set $\partial\mathcal{D}$.

Because of our subsequent cosmological applications we consider dynamical systems with the Hamiltonian function (3), $\mathcal{M} = \mathbb{R}^n$ on the level $H = 0$. Consequently, trajectories of our system are situated in the domain

$$\Omega = \{(q^\alpha, \dot{q}^\alpha) \in \mathbb{R}^{2n} : g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = -2V(q)\}. \tag{6}$$

In the tangent space $T_q(\mathbb{R}^n)$, $q \in \mathbb{R}^n$, we distinguish the three classes of vectors. A vector v is timelike, spacelike or null if $g_{\alpha\beta}v^\alpha v^\beta < 0$, $g_{\alpha\beta}v^\alpha v^\beta > 0$ or $g_{\alpha\beta}v^\alpha v^\beta = 0$, respectively.

In the configuration space we distinguish the following subsets:

$$\begin{aligned}
\mathcal{D}_+ &= \{q \in \mathbb{R}^n : V(q) > 0\} \\
\mathcal{D}_- &= \{q \in \mathbb{R}^n : V(q) < 0\} \\
\partial\mathcal{D} &= \{q \in \mathbb{R}^n : V(q) = 0\}.
\end{aligned}$$

Evidently, $\partial\mathcal{D}$ is a closed boundary set of $\mathcal{D}_\pm$. From the equation defining the level Ω it follows that the type of a tangent vector $v \in T_q\mathcal{M}$ in the distinguished domains is strictly determined, namely,

- if $v \in T_q\mathcal{M}$ and $q \in \mathcal{D}_+$, v is timelike;
- if $v \in T_q\mathcal{M}$ and $q \in \mathcal{D}_-$, v is spacelike;
- if $v \in T_q\mathcal{M}$ and $q \in \partial\mathcal{D}$, v is null.

We can see that a trajectory crossing the boundary changes the domain, say, $\mathcal{D}_+$ into $\mathcal{D}_-$ and the tangent vector to the trajectory at the $q \in \partial\mathcal{D}$ is situated on a cone determined by the kinetic energy form.

The SIMS in a natural way originates from the dynamics of a system in general relativity and cosmology [6, 7, 10–12]. The theory of these systems is still in *statu nascendi*.

It seems that it is of great importance to detect and understand complexity in the dynamical behaviour of such systems. Hamiltonian systems with two degrees of freedom are the simplest problems of general relativity and cosmology dynamics with nontrivial behaviour. The equations of motion of such systems are, in general, nonlinear and coupled in such a way that they are not solvable by standard mathematical techniques. In the generic cases, they are not Liouville integrable and there exist large regions in phase space where chaotic behaviour appears. They are, however, of considerable physical interest as they have often been used to model dynamics of general relativity and cosmology (see table 1). However, there are many controversies around this subject. These disputes are connected with the numerical character of the obtained results, and, as we understand, they were caused by some conceptual problems.

The general theorems of Hofer and Toland can be simply adopted to our case. Then under quite natural hypotheses about the behaviour of the potential function V the existence of of homoclinic, heteroclinic and periodic orbits on a fixed energy surface can be shown.

To explain the significance of the Hofer–Toland results it is useful to translate the theorems in such a way that they correspond to this case.

The quite natural hypotheses are about a kinetic energy form and the behaviour of a potential function.

(A) The eigenvalues of metric $g_{\alpha\beta}$, $(\alpha, \beta = 1, 2)$ are $\lambda_1 < 0 < \lambda_2$, and a quadratic form $g_{\alpha\beta} y^\alpha y^\beta$ is not degenerate.

(B) The potential function $V \in C^\infty$ and there is a closed, bounded convex set $C \subset \mathbb{R}^n$ which is a closure of a component of a set $q \in \mathbb{R}^n : \; V(q) > 0$ and which has the following property: if $q \in \partial C$ and $V'(q) \neq 0$ and $g^{\alpha\beta} \frac{\partial V}{\partial q^\alpha} \frac{\partial V}{\partial q^\beta} = \|\mathrm{grad} V\|^2 = 0$ then $\frac{\partial^2 V}{\partial q^\beta \partial q^\gamma} g^{\alpha\beta} \frac{\partial V}{\partial q^\alpha} g^{\gamma\delta} \frac{\partial V}{\partial q^\delta} < 0$. Moreover, if $q \in \partial C$ and $V'(q) = 0$ then $D_+ \setminus \{q\} \subset q \cup C$ and $V(q) > 0$ for all $q \in \mathrm{Int}(C)$.

It is worth noting that if ∂C is strictly convex then the first part of the above assumption is automatically satisfied provided that $V > 0$ in the interior of C. The second part of the hypothesis only concerns the location of C relative to the set D_+ when there are rest points $V'(q) = 0$ on ∂C.

3. Cosmological background

Our cosmological example assumes a Friedman–Robertson–Walker geometry, i.e. a line element is of the form

$$\mathrm{d}s^2 = a^2(\eta)[-\,\mathrm{d}\eta^2 + \,\mathrm{d}\chi^2 + f^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)]$$

where

$$f(\chi) = \begin{cases} \sin\chi & 0 \leqslant \chi \leqslant \pi \qquad k = 1 \\ \chi & 0 \leqslant \chi < \infty \qquad k = 0 \\ \sinh\chi & 0 \leqslant \chi < \infty \qquad k = -1 \end{cases}$$

and $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \pi$ and η represents 'conformal' time.

The gravitational dynamics is described by the Einstein–Hilbert action

$$S_g = m_{\mathrm{p}}^2 \int \mathrm{d}^4x\,\sqrt{-g}(R - 2\Lambda)$$

where

$$\sqrt{-g} = a^4 f^2(\chi)\sin\theta$$

and the Ricci scalar

$$R = 6\left[\frac{\ddot{a}}{a^3} + \frac{k}{a^2}\right]$$

where the dot represents differentiation with respect to η and m_{p} is the Planck mass. For simplicity we assume that $m_{\mathrm{p}} = \sqrt{1/(12v)}$ with v the conformal volume of the spatial hyperspace.

The action for a conformally coupled massive real self-interacting scalar field is given by

$$S_\phi = -\tfrac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}[\partial_\mu\Phi\partial^\mu\Phi + 2V(\Phi) + \xi R\Phi^2]$$

where $\xi = \frac{1}{6}$ for conformal coupling between the field and gravity.

The dynamical equations can be obtained from the variational principle $\delta(S_g + S_m) = 0$. After dropping the full derivatives with respect to η from the Lagrangian function we finally obtain

$$\mathcal{L} = \frac{1}{2}\left[-\dot{a}^2 + \dot{\psi}^2 + k(a^2 - \psi^2) - \frac{\bar{\Lambda}}{2}a^4 - 2a^4 v V\left(\frac{\psi}{a\sqrt{v}}\right)\right]$$

where $\psi = \Phi a\sqrt{v}$ and $\bar{\Lambda} = \frac{2}{3}\Lambda$.

If we substitute the potential $V(\Phi) = \frac{1}{2}m^2\Phi^2 + \frac{1}{4}\bar{\lambda}\Phi^4$ and denote $\lambda = \bar{\lambda}/v$ we obtain

$$\mathcal{L} = \frac{1}{2}\left[-\dot{a}^2 + \dot{\psi}^2 + k(a^2 - \psi^2) - \frac{\bar{\Lambda}}{2}a^4 - m^2\psi^2 a^2 - \frac{\bar{\lambda}}{2}\psi^4\right]. \tag{7}$$

4. Direct construction of integrable systems. Integrals linear and quadratic in momenta

There is no systematic way to prove integrability for a planar Hamiltonian system of the form (3). However, in order to select integrable systems, we use direct approach. Assuming that a second integral of motion of a prescribed form exists, we can find all potentials which admit such an integral. For a system with Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 - p_y^2) + V(x, y) \tag{8}$$

this is straightforward and the simplest way to apply a known result. We demonstrate it starting from finding all the possible potentials which admit the second integral of motion linear in momenta and having the form

$$I = A(x, y)p_x + B(x, y)p_y. \tag{9}$$

Note that terms of zero order in the momenta have been omitted so that I has a good time parity. We have

$$[I, H] = A_x p_x^2 + (B_x - A_y)p_x p_y - B_y p_y^2 - (AV_x + BV_y) = 0 \tag{10}$$

where subscripts denote partial derivatives and $[\cdot, \cdot]$ denote Poisson brackets.

Since equation (10) must hold identically the following relations are fulfilled:

$$A_x = 0 \tag{11}$$

$$B_x = A_y \tag{12}$$

$$B_y = 0 \tag{13}$$

$$AV_x + BV_y = 0. \tag{14}$$

The general solution of system (11)–(13) is

$$A = \alpha y + \gamma \tag{15}$$

$$B = \alpha x + \beta \tag{16}$$

where α, β and γ are constants.

Two cases of different values of α must be considered. First, put $\alpha = 0$. Then, equation (14) becomes $\gamma V_x + \beta V_y = 0$ with the solution $V = V(\beta x - \gamma y)$. After rotation in hyperbolic space $(x, \mathrm{i}y)$ with $\beta = \cosh\alpha$ and $\gamma = \sinh\alpha$

$$x \to (\beta x + \gamma y)/(\beta^2 - \gamma^2) \equiv X$$

$$y \to (\gamma x + \beta y)/(\beta^2 - \gamma^2) \equiv Y$$

which is formally equivalent to putting $\gamma = 0$, the potential becomes

$$V = V(X)$$

and the corresponding integral is

$$I = p_Y.$$

The second case is $\alpha \neq 0$. Without loss of generality we can assume that $\alpha = 1$ and then we perform the translation

$$x \to x + \beta = X$$

$$y \to y + \gamma = Y$$

which is equivalent to putting $\beta = \gamma = 0$. Equation (14) becomes now

$$YV_X + XV_Y = 0$$

with the solution

$$V = V(X^2 - Y^2) = V(r)$$

where r is a radius in two-dimensional Minkowski space with the metric $\mathrm{d}s^2 = \mathrm{d}x^2 - \mathrm{d}y^2$, $x = r\cosh\alpha$, $y = r\sinh\alpha$.

The corresponding integral is

$$I = YP_x + XP_y = y\dot{x} - x\dot{y}. \tag{17}$$

We conclude that the planar potentials which admit a second integral of motion, which is linear in momenta or can be reduced to this form by means of linear point transformations, depend on one variable or are central potentials in the Minkowski space. The second integral of motion is linear or angular momentum, respectively.

Now, the next step is to search for planar potentials $V(x, y)$ which admit an integral quadratic in momenta

$$I = Ap_x^2 + Bp_xp_y + Cp_y^2 + D \tag{18}$$

where A, B, C and D are functions of x and y. Again the terms which are linear in momenta are omitted in order that I possesses a good time parity.

Darboux [8] obtained a general differential equation for the potential of a simple classical mechanical system admitting an integral of motion I of the form (18). The whole class of such integrable potentials was discovered independently by Dorizzi *et al* [13].

From condition $[H, I] = 0$ for Hamiltonian (8) and integral (18) we obtain the following equations:

$$\begin{aligned} &A_x = 0 \qquad B_x - A_y = 0 \\ &C_x - B_y = 0 \qquad C_y = 0 \end{aligned} \tag{19}$$

$$\begin{aligned} D_x &= 2AV_x + BV_y \\ D_y &= -BV_x - 2CV_y. \end{aligned} \tag{20}$$

The general solution of (19) is

$$\begin{aligned} A &= \alpha y^2 + \beta y + \gamma \\ B &= 2\alpha xy + \beta x + \delta y + \epsilon \\ C &= \alpha x^2 + \delta x + \xi \end{aligned} \tag{21}$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ and ξ are constants while integrability conditions on (20) yield the equation

$$2(A + C)V_{xy} + B(V_{xx} + V_{yy}) + (2A_y + B_x)V_x + (B_y + 2C_x)V_y = 0 \tag{22}$$

where A, B and C are given by (21).

Equation (22) can be treated as a counterpart of the classical Darboux equation. It is a necessary and sufficient condition for a potential $V(x, y)$ to admit a second integral of motion which is quadratic in momenta.

The classical cases examined by Darboux and Whittaker correspond to $\alpha \neq 0$. In this case we may put $\beta = \delta = 0$ by performing an adequate translation and then, if necessary, we can perform a rotation in order to achieve $\epsilon = 0$. We may also put $\xi = 0$ by subtracting from I a suitable amount of H. After some manipulations in our case we obtain for $\alpha = 1$

$$xy(V_{xx} + V_{yy}) + (y^2 + x^2 + \gamma)V_{xy} + 3yV_x + 3xV_y = 0. \tag{23}$$

The full analysis of solutions of equation (22) requires consideration of all possible cases for (a) $\alpha \neq 0$ and (b) $\alpha = 0$.

4.1. The case of $\alpha \neq 0$ and $\gamma \neq 0$

In this case it would be useful to perform the following transformations:

$$x = \frac{uv}{\sqrt{\gamma}} \qquad y = -\frac{\sqrt{(u^2-\gamma)(v^2-\gamma)}}{\sqrt{\gamma}}.$$

Then equation (23) assumes the form

$$(v^2 - u^2)(V_{uv} + 2vV_u - 2uV_v) = 0$$

or

$$((v^2 - u^2)V)_{uv} = 0$$

which has the following solutions:

$$V = \frac{f(u) - g(v)}{u^2 - v^2}$$

where f and g are arbitrary functions of their arguments.

The corresponding second integral of motion can be obtained after solution of (2): it takes the form

$$I = (yp_x + xp_y)^2 + \gamma p_x^2 + \frac{2[v^2 f(u) - u^2 g(v)]}{u^2 - v^2}.$$

4.2. The case of $\alpha \neq 0$ and $\gamma = 0$

In this case we transform equation (23) to new variables $\xi = x^2 - y^2$ and $\eta = \mathrm{i}x/y$ and it becomes

$$\xi V_{\xi\eta} + V_\eta = 0$$

which can immediately be integrated to yield

$$V = f(\xi) + \xi^{-1} g(\eta)$$

where f and g are arbitrary functions.

The second integral of motion has the form

$$I = (xp_y + yp_x)^2 + 2g(\theta)$$

where ρ and θ are coordinates in hyperbolic space.

Let us note that for $g = 0$ the potential V is central in the Minkowski space and I is merely the angular momentum integral. On the other hand if $f = 0$ then potential V takes the form $V = \rho^{-2} g(\theta)$, which is the general form of a homogeneous function of degree two.

4.3. The case of $\alpha^2 = 0$ and $\beta^2 + \delta^2 \neq 0$

In this case by an adequate rotation we may put $\beta = 1$ and $\delta = 0$ and after subtracting H from I we may put $\xi = 0$. Then by translating we may take $\gamma = \epsilon = 0$. Finally equation (22) takes the form

$$2yV_{xy} + 3V_x + x(V_{xx} + V_{yy}) = 0$$

and after transformation to the new variables $\rho = (x^2 - y^2)^{1/2}$, $\eta = \mathrm{i}y$ and $U = \rho V$ the above equation takes the form

$$U_{\rho\rho} - U_{\eta\eta} = 0$$

which may be solved and we obtain

$$V = [f(\rho + \eta) + g(\rho - \eta)]/\rho.$$

The corresponding first integral has the form

$$I = yp_x^2 + xp_xp_y - [(\rho - \eta)f(\rho + \eta) + (\rho + \eta)g(\rho - \eta)]/\rho.$$

4.4. The case of $\alpha = \beta = \delta = 0$

By adding adequate amounts of the first integral H and rotating we may obtain $\xi = 0$, $\gamma = 1$ and $\epsilon = 0$ and then we obtain from (22)

$$V_{xy} = 0$$

so $V = f(x) + g(y)$ and then $I = p_x^2 + 2f(x)$.

Therefore, as was demonstrated, all integrable planar potentials $V(x, y)$ which possess a second integral of motion linear or quadratic in the momenta are known. In the first case the potentials are simply one dimensional or central while in the second case they belong to the four classes of solutions of the equation which is the indefinite counterpart of the Darboux equation. It would be interesting to examine whether these potentials are also separable as is the case for positive definite Hamiltonians. Note that the Darboux equation for definite Hamiltonian systems can be obtained from equation (23) after the substitution $y = -\mathrm{i}\bar{y}$ and $\bar{y}$ satisfies the classical Darboux equation.

5. Nonintegrability of the FRW evolution with scalar fields

In this section we apply Ziglin theory [37, 38] and its extension [22] for proving the nonintegrability of the FRW dynamical system. This approach is attractive because Ziglin theory in its general formulation is applicable for a Hamiltonian system with the Hamiltonian function having an arbitrary form. However, for an effective application of this theory one needs to determine the monodromy group of (normal) variational equations associated with a particular solution. This can be done only for very special cases, and, because of this, several effective formulations of nonintegrability theorems were formulated for systems with a prescribed form of the Hamiltonian (e.g. [18–20, 32–35]). In fact these special formulations consider natural systems with constant positive definite and diagonal forms of the kinetic energy. An exception is the model considered in [27].

5.1. Outline of Ziglin theory

The fundamental papers of Ziglin [37, 38] gave the formulation of a very basic theorem concerning the nonintegrability of analytic Hamiltonian systems. The idea of the Ziglin approach lies in a deep connection between properties of solutions on a complex time plane and the existence of the first integral. This idea has its origins in the works of S W Kovalevskaya and A M Lapunov. Ziglin's works have found many continuations and many important applications [2, 4, 14, 15, 18–20, 25, 26, 32–35].

The main difficulty with the application of the Ziglin theorem is the determination of the monodromy group of NVE. Only in very special cases can we do this analytically (see cited papers). Yoshida [31–36] developed the Ziglin approach for special cases when the Hamiltonian of a system has a natural form and the potential is a homogeneous function. In this case we can find a particular solution in the form of a 'straight-line solution' and its normal variational equations can be transformed to a product of certain copies of hyper-geometric equations for which the monodromy group is known. This allows us to formulate adequate theorems in the form of an algorithm. Below we describe this for a case of a Hamiltonian system with two degrees of freedom.

Consider the Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2) \qquad (q_1, q_2, p_1, p_2) \in \mathbb{C}^4 \tag{24}$$

where $V(q_1, q_2)$ is a homogeneous function of degree k, i.e.

$$V(Cq_1, Cq_2) = C^k V(q_1, q_2). \tag{25}$$

In a generic case this system has straight-line solutions of the form

$$q_1 = C_1\phi(t) \qquad q_2 = C_2\phi(t) \tag{26}$$

where $\phi(t)$ is a solution of a nonlinear equation

$$\ddot{\phi} = -\phi^{k-1}$$

and $(C_1, C_2) \neq (0, 0)$ are solutions of the following system:

$$C_1 = \partial_1 V(C_1, C_2) \qquad C_2 = \partial_2 V(C_1, C_2). \tag{27}$$

The variational equations take the form

$$\begin{bmatrix} \ddot{\xi} \\ \ddot{\eta} \end{bmatrix} = -\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}(\phi(t))^{k-2}$$

where $V_{ij} = \partial_i\partial_j V(C_1, C_2)$ for $i, j = 1, 2$. Since the Hessian of V is symmetric it is diagonalizable by an orthogonal transformation and the system separates to

$$\ddot{\xi} = -\lambda_1\Phi^{k-2}(t)\xi \tag{28}$$

$$\ddot{\eta} = -\lambda_2\Phi^{k-2}(t)\eta \tag{29}$$

where λ_1 and λ_2 are real eigenvalues of the Hessian. Let us note that this is not true for indefinite systems where the Hessian is not a symmetric matrix.

It can be shown that the Hessian of V at $C = (C_1, C_2)$ has the eigenvalue $\lambda_1 = k - 1$. Thus, its second eigenvalue is given by $\lambda := \lambda_2 = \operatorname{tr} V(C_1, C_2) - (k - 1)$, and it is called the integrability index. Equation (29) is a normal variational equation. It can be transformed to the hyper-geometric equation. Monodromy matrices of this equation are parametrized by λ and conditions of the Ziglin theorem put restrictions on the values of λ—simply, we can identify those values of λ for which our system is not integrable (more precisely, does not possess an additional meromorphic first integral). To state it accurately let us define

$$I_k(p) = \left[\frac{k}{2}p(p+1) - p, \frac{k}{2}p(p+1) + p + 1\right] \qquad p \in \mathbb{N} \tag{30}$$

and

$$N_k = \mathbb{R} \setminus \bigcup_{p\in\mathbb{N}} I_k(p). \tag{31}$$

Then it follows that the Hamiltonian system (24) with homogeneous potential (25) of degree k is not integrable if the integrability index λ corresponding to a certain straight-line solution (26) belongs to N_k. Let us note that usually equations (27) have several solutions and thus we have to check the Yoshida criterion for every one of them.

5.2. Application to the Friedman–Robertson–Walker Hamiltonian system

Let us consider the Friedman–Robertson–Walker (FRW) system defined by the following Hamiltonian:

$$H = \tfrac{1}{2}(-p_1^2 + p_2^2) + V(q_1, q_2) \tag{32}$$

where

$$V(q_1, q_2) = \frac{1}{2}\left[-kq_1^2 + kq_2^2 + \frac{\Lambda}{2}q_1^4 + \mu q_1^2 q_2^2 + \frac{\lambda}{4}q_2^4\right]$$

and $(k, \Lambda, \lambda, \mu) \in \mathbb{R}^4$ are parameters of the problem. This Hamiltonian corresponds to the Lagrangian function (7) derived in section 3 (we denote here $\bar{\Lambda} = \Lambda$ and $\mu = m^2$).

It is obvious that there exist two planes

$$\Pi_k = \{(q_1, q_2, p_1, p_2) \in \mathbb{C}^4 : p_k = 0 \wedge q_k = 0\} \qquad k = 1, 2$$

which are invariant with respect to flow generated by H. We show that there exists a third invariant plane given by

$$\Pi_3 = \{(q_1, q_2, p_1, p_2) \in \mathbb{C}^4 : q_2 = \alpha q_1 \wedge p_2 = -\alpha p_1\}$$

where $\alpha \in \mathbb{C}$ is a certain constant depending on parameters (Λ, λ, μ). In fact, from $q_1 = \alpha q_2$ and the equations of motion we obtain that the following equation is satisfied:

$$-\partial_2 V = \alpha \partial_1 V.$$

This gives

$$[(\mu + \Lambda) + (\mu + \lambda)\alpha^2]q_1^3 = 0$$

and thus

$$\alpha^2 = -\frac{\mu + \Lambda}{\mu + \lambda}.$$

The FRW system restricted to the plane Π_k reduces to a system with one degree of freedom and thus it can be investigated analytically. In particular we can find its nontrivial solutions lying on these planes. However, for normal variational equations corresponding to these we are not able to determine the monodromy matrices. The case of $k \neq 0$ is more complicated and detailed analysis of these equations and a proof of nonintegrability is not presented here.

In what follows we restrict ourselves to the case $k = 0$. In such a case we have a Hamiltonian system with a homogeneous potential and for this system we can apply the Yoshida criterion. To this end let as make the following complex canonical transformation: $(p_1, q_1) \rightarrow (\mathrm{i}p_1, -\mathrm{i}q_1)$. Let us note that such a trick gives rise to the application of Yoshida theorems only if the potential function is homogeneous, of degree $2k$. After this transformation the Hamiltonian function has the form

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2) \tag{33}$$

where

$$V(q_1, q_2) = \frac{\Lambda}{4}q_1^4 - \frac{\mu}{2}q_1^2 q_2^2 + \frac{\lambda}{2}q_2^4.$$

Equation

$$q = V'(q) \qquad q = (q_1, q_2)$$

has the following solutions:

$$z_1 = (\pm\lambda^{-1/2}, 0) \qquad z_2 = (0, \pm\lambda^{-1/2}) \qquad z_3 = \left(\pm\sqrt{\frac{\lambda + \mu}{\Lambda\lambda - \mu^2}}, \pm\sqrt{\frac{\lambda + \mu}{\Lambda\lambda - \mu^2}}\right).$$

The integrability indices for these points are

$$\lambda_i = -\operatorname{tr} V''(z_i) - 3 \qquad i = 1, 2, 3$$

and

$$\lambda_1 = -\frac{\mu}{\Lambda} \qquad \lambda_2 = -\frac{\mu}{\lambda} \qquad \lambda_3 = \frac{\lambda_1\lambda_2 - 2(\lambda_1 + \lambda_2) + 3}{1 - \lambda_1\lambda_2}. \tag{34}$$

Thus, from the Yoshida criterion it follows that if there is $l \in \{1, 2, 3\}$ such that $\lambda_l \in N_4$ then system (32) has no additional meromorphic first integral that is functionally independent of H.

We show now that complex canonical transformation performed before application of the Yoshida algorithm is in fact unnecessary. In fact, let us consider the system with the Hamiltonian

$$H = \tfrac{1}{2}(-p_1^2 + p_2^2) + V(q_1, q_2) \tag{35}$$

where

$$V(q_1, q_2) = \frac{\Lambda}{4}q_1^4 + \frac{\mu}{2}q_1^2 q_2^2 + \frac{\lambda}{4}q_2^4.$$

We look for 'straight-line' solutions $z(t) = (\phi c_1, \phi c_2, -\dot{\phi} c_1, \dot{\phi} c_2)$ where $\phi = \phi(t)$. As it is easy to see, $z(t)$ is a solution of the system generated by (35) if and only if $c = (c_1, c_2)$ is a solution of the following system:

$$c_1 = -\partial_1 V(c_1, c_2) \qquad c_2 = \partial_1 V(c_1, c_2) \tag{36}$$

and ϕ satisfies the following differential equation:

$$\ddot{\phi} + \phi^3 = 0.$$

Equations (36) have three types of solution,

$$c_{(1)} = (\pm \mathrm{i}\Lambda^{-1/2}, 0) \qquad c_{(2)} = (0, \pm\lambda^{-1/2})$$

and

$$c_{(3)} = \left(\pm\sqrt{\frac{\mu - \lambda}{\Lambda\lambda + \mu^2}}, \pm\sqrt{\frac{\Lambda + \mu}{\Lambda\lambda + \mu^2}}\right).$$

The normal variational equations corresponding to $c_{(1)}$ and $c_{(2)}$ have the forms

$$\ddot{\eta} = \frac{\mu}{\Lambda}\phi(t)^2\eta \qquad \ddot{\eta} = \frac{\mu}{\lambda}\phi(t)^2\eta$$

respectively. Thus, the corresponding Yoshida integrability coefficients are

$$\lambda_1 = -\frac{\mu}{\Lambda} \qquad \lambda_2 = -\frac{\mu}{\lambda}.$$

In the case of solution $c_{(3)}$ we can either use a hyperbolic rotation to locate it along one axis or we can choose noncanonical variables for the variational equations. After that we obtain the third integrabilty index λ_3 of the form (34).

We mention the above constructions for the following reason. In the general settings, in Ziglin theory the system is considered in a complex phase space and with complex time. If we are able to prove nonintegrability we show the nonexistence of the complex meromorphic first integral. Of course, what we need is to prove the nonexistence of the real first integral. As shown recently by Ziglin [39], this is possible in certain situations when we are able to control loops generating the monodromy matrices. At the same time it is important to locate a particular solution in the real part of the phase space.

5.3. Application of the Morales–Ramis theory

Recent results of Morales-Ruiz and Ramis (see book [22]) extend Ziglin theory by connecting it with the differential Galois theory. Results obtained until now (see the cited book) are not only important from a theoretical point of view but also give a very strong tool to study applied problems.

In this approach we restrict ourselves to proving complete Liouvillian nonintegrability but instead of using the monodromy group we investigate the differential Galois group of variational equations. Usually the Galois group is bigger than the monodromy group and because of this one can achieve the nonintegrability result more easily. Lack of space does not allow us to present the basic idea of this interesting and very effective approach. Here we mention only the result of application of this theory to a case considered by Yoshida, i.e. for natural Hamiltonian systems with a homogeneous potential of degree k. For such a situation Morales-Ruiz and Ramis proved the following. Let λ_i be an eigenvalue (different from $k-1$) of the Hessian of the potential evaluated at a point which corresponds to a straight-line solution and k the degree of homogeneity of the potential. Then, if the system is integrable (in the Liouville sense) (λ_i, k) (for all straight-line solutions) belongs to a certain fully described discrete set. (For details see theorem 5.1 in [22].) This gives a stronger result than the Yoshida criterion because the Yoshida criterion implies that for the same assumptions λ_i belongs to set $\bigcup_{p\in\mathbb{N}} I_k(p)$, but this set has a nonempty interior.

Application of the theorem cited above to the FRW Hamiltonian system (35) gives the following result. Let us introduce the following three discrete sets:

$$
\begin{aligned}
I_1 &= \{p(2p-1) | p \in \mathbb{Z}\} \\
I_2 &= \{\tfrac{1}{8}[-1 + 16(\tfrac{1}{3} + p)^2] | p \in \mathbb{Z}\} \\
I_3 &= \{\tfrac{1}{2}[\tfrac{3}{4} + 4p(p-1)] | p \in \mathbb{Z}\}.
\end{aligned}
$$

Then if $\{\lambda_1, \lambda_2, \lambda_3\} \not\subset I = I_1 \cup I_2 \cup I_3$ the system is nonintegrable. Here $\{\lambda_1, \lambda_2, \lambda_3\}$ are given by (34).

It is interesting to select those cases when $\{\lambda_1, \lambda_2, \lambda_3\} \subset I$, i.e. those values of parameters for which the system can be integrable. Note that λ_3 is a symmetric function of λ_1 and λ_2, thus we specify such a case that it is enough to know (λ_1, λ_2). It is easy to observe that if $\lambda_1 = 1$ or $\lambda_2 = 1$ then $\lambda_3 = 1$. Assume for example that $\lambda_2 = \lambda_3 = 1$. Then $\mu = -\lambda$ and $\lambda_1 = \lambda/\Lambda$. If the system is integrable then $\lambda/\Lambda \subset I$. Although set I is discrete it is difficult to test whether for $\lambda/\Lambda \subset I$ the system is integrable or not.

6. Conclusions

The aim of this paper was to show the need to develop methods which allow us to study systems with indefinite kinetic energy form. We mention some open problems and, as an illustration, we consider the simplest two-dimensional dynamical system describing the evolution of FRW models with a scalar field. The corresponding dynamical system can be reduced to the Hamiltonian form with indefinite kinetic energy form and polynomial potential. Physically meaningful trajectories of this system belong to the hyper-surface determined by the Hamiltonian constraint $H = 0$. We have presented examples showing the effectiveness of using (a) the direct method of construction of linear and quadratic first integrals; (b) the Ziglin and Yoshida theorems concerning nonintegrability of Hamiltonian systems with homogeneous potential functions; (c) the Morales–Ramis theorem on nonintegrability of Hamiltonian systems with complex potential functions.

The systems under consideration are analysed in the literature and complex behaviour of trajectories was demonstrated by different methods. However the main aim of our paper was devoted to the analysis of nonintegrability of simple indefinite mechanical systems in general. We think that the first step in this direction has been made.

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